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Comparative Study of Robot Kinematic Calibration Algorithms using a Unified Geometric Framework*

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***Abstract*— In this paper, we conduct a comparative study of three well known robot kinematic calibration algorithms, namely the *Denavit-Hartenberg* (DH) parameter algorithm, the *product of exponentials* (POE) algorithm, and the *local POE* (LPOE) algorithm. To cope with distinct formulations associated to different algorithms, we propose a unified geometric framework which is based on POE kinematics and a novel Adjoint error model. The Adjoint error model offers us an extremely efficient way to benchmark the aforesaid calibration algorithms, and also compare them to a novel calibration algorithm based on the Adjoint error model.**

I. INTRODUCTION

A. Problem statement

The proliferation of industrial robots in the computer/communication/consumer electronics (3C) manufacturing industry calls for more capable robots that match the precision requirement and complexity of typical 3C assembly tasks. Robust and efficient robot calibration technology is the key to bringing an average industrial robots to this challenging new scenario.

Robot calibration involve three levels of activities [1]: the joint level calibration, kinematic (geometric) calibration and the non-kinematic (non-geometric) calibration. Since the geometric error of robot manipulators accounts for over ninety-five percent of overall error [2], kinematic calibration becomes a major means to improve robot precision. Great efforts have been put into developing a robust and efficient kinematic calibration algorithm [1,3]–[5]. The robustness refers to it capability of handling exceptions in robot kinematic parametrization and error modeling.

The robustness and efficiency of calibration algorithms are of crucial importance to its successful implementation in 3C robotics, where automatic calibration and re-calibration is taking the future trend. It becomes necessary to reconsider the aforementioned calibration algorithms in a comparative manner. This seemingly simple task is hindered by the essential differences among the various calibration algorithms, in particular in kinematic parametrization, error modeling and error update methods.

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There are two types of kinematics parametrization for robot manipulators: the *Denavit-Hartenberg (D-H)* convention [6,7] and the *product of exponentials (POE)* convention [8,9]. D-H convention [6] is based on parametrization of homogeneous matrix between consecutive coordinate frames that are attached to each link of the robot, including the base and the end-effector. The POE convention [9] is based on parametrization of each joint axis in its initial configuration, and with respect to a single reference frame. While D-H based calibration algorithms update the D-H parameters directly, POE based algorithms update the joint twists instead.

This paper conducts a comparative study of three well known calibration algorithms in a unified geometric framework. First, we introduce the concept of *quotient space* to give a unified comparison of the robustness of different parametrization conventions, which are in terms of completeness, minimality and parametric continuity [10]. Then, we propose the *Adjoint error model* for a unified treatment of the error models used in three calibration algorithms, namely the *Denavit-Hartenberg* (DH) parameter algorithm, the *product of exponentials* (POE) algorithm, and the *local POE* (LPOE) algorithm. We also propose a fourth calibration algorithm based on the Adjoint error model. The unified treatment of the four calibration algorithms finally leads to an extremely easy method to compare their efficiency by counting the number of basic transformations involved in their formulation. Side issues such as identifiability and parameter redundancy elimination are also discussed, with details found in a sequel to this paper [11].

B. Organization of the Paper

This paper is organized as follows: in Section II, we give a minimum review of the mathematical tools and notational conventions needed for both the rest of and the sequel [11] to this paper; in Section III, we propose a unified geometric framework for error modeling; in Section IV, we conduct a comparative study of four calibration algorithms in terms of robustness and efficiency. In particular, we propose a novel kinematic error model and an associated calibration algorithm called the *Adjoint error* algorithm. Finally we give our conclusion in Section V.

II. MATHEMATICAL PRELIMINARIES AND NOTATIONAL CONVENTIONS

A. Describing Rigid Motion of Body and Point Using the ***Lie Group of Rigid Displacement*** $SE(3)$ *(basically follows [9])*

The rigid displacement of coordinate frame B w.r.t. another A can be completely described by a 4×4 homogeneous matrix:

$$g_{ab} = \begin{bmatrix} R_{ab} & p_{ab} \\ 0_{1\times 3} & 1 \end{bmatrix} \in \mathbb{R}^{4\times 4}. \quad (1)$$

where p_{ab} is the position vector of o_B, the origin of frame B, w.r.t. frame A; the unit coordinate axes $\mathbf{x}_{ab}, \mathbf{y}_{ab}, \mathbf{z}_{ab}$ of B w.r.t. A comprise the 3 columns of the orthogonal matrix R_{ab}. The set of all such matrices form a *Lie group* under matrix multiplication, known as the 6-D special Euclidean group of $\mathbb{R}^3$, and is denoted by $SE(3)$:

$$SE(3) \triangleq \left\{ \begin{bmatrix} R & p \\ 0_{1\times 3} & 1 \end{bmatrix} \in \mathbb{R}^{4\times 4} \middle| R \in SO(3), p \in \mathbb{R}^3 \right\}. \quad (2)$$

When B is attached to a moving body and A is attached to a reference body, the rigid motion of $B(t), t \in \mathbb{R}$ w.r.t. A is given by $g_a(t) \triangleq g_{ab}(t) \cdot g_{ab}^{-1}(0), t \in \mathbb{R}$. If, on the other hand, the same motion is observed from B, we have $g_b(t) \triangleq g_{ab}^{-1}(0) \cdot g_{ab}(t)$. The two motions are related by the *conjugation map* $I_{g_{ab}(0)}(\bullet) \triangleq g_{ab}(0) \cdot (\bullet) \cdot g_{ab}^{-1}(0)$.

B. Describing Velocity Using the **Lie Algebra** $se(3)$ of $SE(3)$ (basically follows [9,12])

The velocity of a rigid motion $g_{ab}(t)$, viewed from the frame A or $B(t)$ is given by:

$$\hat{V}_{ab}^a(t) \triangleq \dot{g}_{ab}(t) \cdot g_{ab}^{-1}(t), \text{ or } \hat{V}_{ab}^b(t) \triangleq g_{ab}^{-1}(t) \cdot \dot{g}_{ab}(t). \quad (3)$$

The corresponding velocity of a point q w.r.t. A or B is given by:

$$\dot{q}_a(t) = \hat{V}_{ab}^a \cdot q_a(t), \text{ or } \dot{q}_b(t) = \hat{V}_{ab}^a \cdot q_b. \quad (4)$$

All velocities take the form of a *twist*,

$$\begin{bmatrix} \hat{\omega} & v \\ 0_{1\times 3} & 0 \end{bmatrix} \in \mathbb{R}^{4\times 4}, \omega, v \in \mathbb{R}^3, \hat{\omega} \cdot (\bullet) \triangleq \omega \times (\bullet). \quad (5)$$

which form a 6-D vector space, known as the *Lie algebra* $se(3)$ of $SE(3)$. It has an additional algebra structure known as the *Lie bracket*, $[\bullet, \bullet]$:

$$[\hat{\xi}_1, \hat{\xi}_2] \triangleq \hat{\xi}_1 \hat{\xi}_2 - \hat{\xi}_2 \hat{\xi}_1. \quad (6)$$

which is anti-symmetrically bilinear. We usually take advantage of the vector isomorphism:

$$\wedge : \xi = \begin{bmatrix} v \\ \omega \end{bmatrix} \mapsto \hat{\xi} = \begin{bmatrix} \hat{\omega} & v \\ 0_{1\times 3} & 0 \end{bmatrix}, v, \omega \in \mathbb{R}^3. \quad (7)$$

to assign a standard basis $\{\hat{e}_i\}_{i=1}^6$ for $se(3)$, where $e_i's$ are the canonical basis for $\mathbb{R}^6$.

$\hat{V}_{ab}^b$ and $\hat{V}_{ab}^a = g_{ab} \hat{V}_{ab}^b g_{ab}^{-1}$ are related by the *Adjoint map* $Ad_g(\bullet)$:

$$\begin{aligned} Ad_g &: \hat{\xi} \mapsto Ad_g \hat{\xi} \triangleq g \cdot \hat{\xi} \cdot g^{-1}, \text{ or} \\ &\xi \mapsto Ad_g \xi = \begin{bmatrix} R & \hat{p}R \\ 0_{3\times 3} & R \end{bmatrix} \cdot \xi. \end{aligned} \quad (8)$$

The Adjoint map is the instantaneous version of the conjugation map I_g, i.e. a change of coordinates for velocity. Alternatively, it can be explained as a rigid displacement of velocity.

At this point, we need to introduce the *adjoint map*, denoted $ad_{\hat{\xi}}(\bullet), \xi = (v^T, \omega^T)^T$:

$$\begin{aligned} ad_{\hat{\xi}} &: \hat{\eta} \mapsto ad_{\hat{\xi}} \hat{\eta} \triangleq [\hat{\xi}, \hat{\eta}], \hat{\xi}, \hat{\eta} \in se(3), \text{ or} \\ &\eta \mapsto ad_\xi \eta, ad_\xi = \begin{bmatrix} \hat{\omega} & \hat{v} \\ 0_{3\times 3} & \hat{\omega} \end{bmatrix} \in \mathbb{R}^{6\times 6}. \end{aligned} \quad (9)$$

Physically, $ad_{\hat{\xi}}$ is the rate of $Ad_{e^{\hat{\xi} t}}$:

$$ad_\xi = \frac{\mathrm{d}}{\mathrm{d}t} \left(Ad_{e^{\hat{\xi} t}} \right) \bigg|_0 . \quad (10)$$

We may also need the following useful equation from [13]:

$$Ad_{e^{\hat{\xi}}} = e^{ad_{\hat{\xi}}}. \quad (11)$$

C. Describing Screw Motion Using the **Exponential Map** $\exp : se(3) \to SE(3)$ (basically follows [9])

The screw motion $g(t)$ about a spatial line passing through point $q \in \mathbb{R}^3$ with unit direction vector $\omega \in \mathbb{R}^3$ and pitch $h \in \mathbb{R}$ is given by:

$$\begin{aligned} g(t) &= \begin{bmatrix} e^{\hat{\omega} t} & (I - e^{\hat{\omega} t}) q + h \omega t \\ 0_{1\times 3} & 1 \end{bmatrix} \\ &= e^{\hat{\xi} t} \in SE(3), \xi = \begin{bmatrix} q \times \omega + h\omega \\ \omega \end{bmatrix}, \|\omega\| = 1. \end{aligned} \quad (12)$$

where $e^{\hat{\omega} t} = I + \hat{\omega} t + \frac{1}{2!} (\hat{\omega} t)^2 + \cdots$ is given by the Rodriguez formula:

$$e^{\hat{\omega} t} = I + \hat{\omega} \sin t + \hat{\omega}^2 (1 - \cos t), \|\omega\| = 1. \quad (13)$$

Any rigid body motion is equivalent to a finite screw motion [9], i.e. the *exponential map* $\exp : se(3) \to SE(3), \hat{\xi} \mapsto e^{\hat{\xi}}$ is surjective. Conversely, any velocity $\xi = (v^T, \omega^T)^T$ is also instantaneously equivalent to that of a screw motion. Thus we entitle a twist $\hat{\xi}$ with the geometrical meaning of a screw ξ. Given a basis $\{\hat{\eta}_i\}_{i=1}^6$ of $se(3)$, exp defines two parametrization for $SE(3)$ [12]:

$$\begin{cases} \text{The 1st-kind} : (\theta_1, \ldots, \theta_6) \in \mathbb{R}^6 \mapsto e^{\sum_{i=1}^6 \hat{\eta}_i \theta_i}, \\ \text{The 2nd-kind} : (\theta_1, \ldots, \theta_6) \in \mathbb{R}^6 \mapsto \prod_{i=1}^6 e^{\hat{\eta}_i \theta_i}. \end{cases} \quad (14)$$

The 2nd-kind is also known as the *product of exponentials (POE)*, which represents the direct kinematics map of a 6-DoF robot manipulator [9].

D. Describing Joint Motion and Axial Symmetry Using **Lie subgroups** of $SE(3)$ and **Quotient Spaces** (basically follows [14,15])

Lie subgroups of $SE(3)$ are subsets that are closed under group multiplication and inversion. They are often involved in describing motion of primitive joints [12] and the symmetry of certain geometric entity [14]. For example, for a constant screw ξ, the set $\{e^{\hat{\xi}\theta} | \theta \in \mathbb{R}\}$ forms a 1-D subgroup of $SE(3)$. It can be explained as the motion set generated by a revolute ($h = 0$), a prismatic ($h = \infty$) or a helicoidal joint ($h \neq 0$), where θ is the joint variable.

Besides, an infinitely long cylinder or a directed line (or a revolute joint axis) can be rotated about and translated along its own axis without altering its configuration. The subset of $g \in SE(3)$ that fixes $\xi = ((q \times \omega)^T, \omega^T)^T$ forms a 2-D

cylindrical subgroup (associated with ξ), denoted by $C(\xi)$:

$$C(\xi) \triangleq \left\{ e^{\hat{\xi}\theta_1 + \hat{V}_\xi \theta_2} | \theta_i \in \mathbb{R} \right\}, V_\xi \triangleq \begin{bmatrix} \omega \\ 0_{3\times 1} \end{bmatrix}. \quad (15)$$

Similarly, the symmetry of a free vector $\xi = (v^T, 0)^T$ (or a prismatic joint axis) is given by the 4-D *Schönflies subgroup* (associated with ξ) [12], denoted by $X(\xi)$:

$$X(\xi) \triangleq \left\{ e^{\sum_{i=1}^{3} \hat{e}_i \theta_i + \hat{\Omega}_\xi \theta_4} | \theta_i \in \mathbb{R} \right\}, \Omega_\xi \triangleq \begin{bmatrix} 0_{3\times 1} \\ v \end{bmatrix}. \quad (16)$$

It can be used to describe pick-and-place motion (SCARA motion). It is well known that the *Lie subalgebra* $\mathfrak{g}$ of a Lie subgroup $G \subset SE(3)$, defined by $\mathfrak{g} := \{\hat{\xi} \in se(3) | e^{\hat{\xi}} \in G\}$, is indeed a Lie algebra since it is closed under $[\bullet, \bullet]$. We shall denote the Lie subalgebra of $C(\xi)$ and $X(\xi)$ by $\mathfrak{c}(\xi)$ and $\mathfrak{x}(\xi)$ respectively:

$$\mathfrak{c}(\xi) = \mathrm{span}\{\hat{\xi}, \hat{V}_\xi\}, \mathfrak{x}(\xi) = \mathrm{span}\{\hat{e}_1, \hat{e}_2, \hat{e}_3, \hat{\Omega}_\xi\}. \quad (17)$$

Due to the existence of symmetry, the arbitrary displacements $Ad_g \xi$ of a joint axis ξ with $g \in SE(3)$ are not in one-to-one correspondence with elements $g \in SE(3)$. The true configuration space of ξ is given by the collection of all left cosets $gC(\xi)$ (or $gX(\xi)$), denoted $SE(3)/C(\xi)$ (or $SE(3)/X(\xi)$):

$$SE(3)/C(\xi) \triangleq \{gC(\xi) | g \in SE(3)\}. \quad (18)$$

$SE(3)/C(\xi)$ $(SE(3)/X(\xi))$ is a 4 (2)-D *quotient space* [15], which can be intuitively understood as what is left off after removing the redundant symmetry information $C(\xi)$ (or $X(\xi)$). The velocity space of $SE(3)/C(\xi)$ can be naturally identified with $se(3)/\mathfrak{c}(\xi)$:

$$se(3)/\mathfrak{c}(\xi) \triangleq \{\hat{\eta} + \mathfrak{c}(\xi) | \hat{\eta} \in se(3)\}. \quad (19)$$

the factor space of all affine subspaces $\hat{\eta} + \mathfrak{c}(\xi)$ in $se(3)$ (see for example [16]). For any subspace $\widehat{W}(\xi) \subset se(3)$ with $\widehat{W}(\xi) \oplus \mathfrak{c}(\xi) = se(3)$, we have the following isomorphism:

$$\pi : \widehat{W}(\xi) \to se(3)/\mathfrak{c}(\xi), \hat{\eta} \mapsto \hat{\eta} + \mathfrak{c}(\xi), \hat{\eta} \in \widehat{W}(\xi). \quad (20)$$

so that given a basis $\{\hat{\eta}_i\}_{i=1}^4$ of $\widehat{W}(\xi)$, we have a natural parametrization for $SE(3)/C(\xi)$:

$$(\theta_1, \ldots, \theta_4) \in \mathbb{R}^4 \mapsto e^{\sum_{i=1}^{4} \hat{\eta}_i \theta_i} C(\xi). \quad (21)$$

and similarly for $SE(3)/X(\xi)$:

$$(\theta_1, \theta_2) \in \mathbb{R}^2 \mapsto e^{\sum_{i=1}^{2} \hat{\eta}_i \theta_i} X(\xi). \quad (22)$$

III. A UNIFIED ERROR MODEL FOR ROBOT KINEMATICS CALIBRATION

A. The Problem of Kinematic Calibration

From now on, we let A be the spatial reference frame S and B be the end-effector tool frame T. For kinematic calibration, we need to consider the generalized direct kinematics (GDK) map of a robot manipulator, which describes the end-effector motion of a robot manipulator as a function of both its joint variables $\theta \in \Theta$ and kinematic parameters $K \in \mathcal{K}$:

$$f : \Theta \times \mathcal{K} \to SE(3), (\theta, K) \mapsto g(\theta, K). \quad (23)$$

When the actual value of θ and K deviate from certain nominal value $\bar{\theta}$ and $\bar{K}$, a deviation of $g(\theta, K)$ from $g(\bar{\theta}, \bar{K})$ can be detected by external measurement devices. The problem of calibration amounts to identifying the joint offset $\Delta\theta = \theta - \bar{\theta}$ and kinematic parameter error $\Delta K = K - \bar{K}$ from M pose error measurements $\delta g^m = g(\theta^m, K) - g(\bar{\theta}^m, \bar{K}), m = 1, \ldots, M$. Usually, a linearization of the problem is used in the identification process and then iterated (indicated by the second superscript k) to converge to the true value:

$$\begin{aligned} \delta g^{m,k} &\approx \left.\frac{\partial g}{\partial \theta}\right|_{\theta^{m,k}} \delta\theta^k + \left.\frac{\partial g}{\partial K}\right|_{K^k} \delta K^k, \\ K^{k+1} &= K^k + \delta K^k, \\ \theta^{m,k+1} &= \Delta\theta^{m,k} + \delta\theta^k, \\ m &= 1, \ldots, M, k \in \mathbb{Z}_+. \end{aligned} \quad (24)$$

Since the dimension of $\Theta \times \mathcal{K}$ is higher than that of $SE(3)$, multiple configurations θ^m are taken to solve for $(\delta\theta^k, \delta K^k)$ in a least square manner. Most calibration algorithms follow this routine and are essentially Newton algorithms.

Take the POE model for example, the GDK is given by [9]:

$$\begin{aligned} f : \underbrace{\mathbb{R}^6}_{\Theta} \times \underbrace{se(3)^6 \times SE(3)}_{\mathcal{K}} &\to SE(3), \\ (\theta_1, \ldots, \theta_6; \xi_1, \ldots, \xi_6; g_{st}(0)) &\mapsto \prod_{i=1}^{6} e^{\hat{\xi}_i \theta_i} \cdot g_{st}(0). \end{aligned} \quad (25)$$

The differential kinematics is given by:

$$\begin{aligned} y = \delta g \cdot g^{-1} \approx \sum_{i=1}^{6} Ad_{i-1} \left(\delta e^{\hat{\xi}_i \theta_i} e^{-\hat{\xi}_i \theta_i} \right) + \\ Ad_6(\delta g_{st} g_{st}^{-1}),\ Ad_i \triangleq \prod_{j=1}^{i} Ad_{e^{\hat{\xi}_j \theta_j}}. \end{aligned} \quad (26)$$

Therefore, it suffices to consider the differential kinematics $\delta e^{\hat{\xi}_i \theta_i} e^{-\hat{\xi}_i \theta_i}$ of a single joint first, and derive (26) through spatial propagations Ad_i's.

B. Differential Error Kinematics of a single Joint

1) The POE approach [4]*:* Given a constant screw ξ, we simply have:

$$\frac{\mathrm{d}}{\mathrm{d}t} e^{\hat{\xi} t} = \hat{\xi} e^{\hat{\xi} t} = e^{\hat{\xi} t} \hat{\xi}. \quad (27)$$

For a parameter varying screw $\xi(t)$, we have [13]:

$$\left(\frac{\mathrm{d}}{\mathrm{d}t} e^{\hat{\xi}(t)} e^{-\hat{\xi}(t)} \right)^\vee = \int_0^1 Ad_{e^{\hat{\xi}(t)s}} \mathrm{d}s \cdot \dot{\hat{\xi}}. \quad (28)$$

where $\vee$ denotes the inverse of $\wedge$ in (7). Therefore we have:

$$\begin{aligned} (\delta e^{\hat{\xi}_i \theta_i} e^{-\hat{\xi}_i \theta_i})^\vee &\approx \int_0^1 Ad_{e^{\hat{\xi}_i \theta_i s}} \mathrm{d}s (\theta_i \delta\xi_i + \xi \delta\theta_i) \\ &= \xi \delta\theta_i + \theta_i \int_0^1 Ad_{e^{\hat{\xi}_i \theta_i s}} \mathrm{d}s \cdot \delta\xi_i. \end{aligned} \quad (29)$$

We refer to $\theta_i \int_0^1 Ad_{e^{\hat{\xi}_i \theta_i s}} \mathrm{d}s$ as the *integral Adjoint map* and denote it by $iAd_{\hat{\xi}_i \theta_i}$. Based on this result, Okamura and Park proposed the POE calibration algorithm [4]. It is pointed out later by the first author [17] and also [18] that $\delta\theta_i$'s and g_{st} cannot be simultaneously identified. In the rest of the paper, all $\delta\theta_i$'s in the POE formulation are ignored without further explanation.

As pointed out in [4,18], since $iAd_{e^{\hat{\xi}\theta}}$ is in general invertible, the error update $\delta\xi$ can take any value and $\xi+\delta\xi$ may not retain a constant pitch h. For $h=0$ (revolute joint), the following quadratic constraints are imposed after each iteration [4,18]:

$$\|\omega_i^k\| = 1,\ \omega_i^{k^T}\cdot v_i^k = 0, i = 1,\ldots,6. \tag{30}$$

As pointed in [17] and explained in Section II, joint axis misalignment can be naturally thought of as an Adjoint error $\delta\hat{\eta}\in se(3)$:

$$\begin{aligned}\delta\xi^k &= Ad_{e^{\delta\hat{\eta}^k}}\xi^k - \xi^k \\ &= e^{ad_{\delta\hat{\eta}^k}}\xi^k - \xi^k \approx ad_{\delta\eta^k}\xi^k = -ad_{\xi^k}\delta\eta^k.\end{aligned} \tag{31}$$

and therefore by chain rule:

$$\delta e^{\hat{\xi}\theta}e^{-\hat{\xi}\theta} \approx -iAd_{e^{\hat{\xi}\theta}}\cdot ad_\xi \cdot \delta\eta. \tag{32}$$

As as result, the *Adjoint error* update:

$$\xi^{k+1} = Ad_{e^{\delta\hat{\eta}^k}}\xi^k. \tag{33}$$

elegantly removes the quadratic constraints (30) used in [4, 18]. (32) can be further reduced to:

$$(\delta e^{\hat{\xi}\theta}e^{-\hat{\xi}\theta})^\vee \approx \left(I - Ad_{e^{\hat{\xi}\theta}}\right)\delta\eta. \tag{34}$$

2) The D-H (modified D-H) approach [19,20]*:* Although the D-H approach involves consecutive transformation of local frames, its model for a single joint can be essentially represented as a product of 4 exponentials:

$$\begin{aligned}g_{i-1,i} &= e^{\hat{e}_6\theta_i}e^{\hat{e}_3 d_i}e^{\hat{e}_4\alpha_i}e^{\hat{e}_1 a_i} \\ &= e^{\hat{e}_6\theta_i+\hat{e}_3 d_i}e^{\hat{e}_4\alpha_i+\hat{e}_1 a_i}.\end{aligned} \tag{35}$$

The second equality in (35) holds since $[\hat{e}_6,\hat{e}_3]=[\hat{e}_4,\hat{e}_1]=0$. Its differential is given by:

$$\begin{aligned}(\delta g_{i-1,i}g_{i-1,i}^{-1})^\vee \approx\ & e_6\delta\theta_i + e_3\delta d_i + \\ & Ad_{e^{\hat{e}_6\theta_i+\hat{e}_3 d_i}}(e_4\delta\alpha_i + e_1\delta a_i).\end{aligned} \tag{36}$$

In other words, the DH approach model avoids the integral Adjoint map $iAd_{(\bullet)}$ at the cost of computing four Adjoint maps.

3) The local POE approach [5]*:* The LPOE approach is a mixture of the D-H approach and the POE approach [5]. It addresses the joint axis misalignment as errors in consecutive frames attached to each link. The choice of local frames is such that each joint axis is locally represented as $\hat{e}_6$:

$$h_{i-1,i} = g_{i-1,i}\cdot e^{\hat{e}_6\theta_i}, g_{i-1,i}^{k+1} = e^{\delta\hat{\eta}_i^k}g_{i-1,i}^k. \tag{37}$$

and therefore:

$$(\delta g_{i-1,i}g_{i-1,i}^{-1})^\vee \approx \delta\eta_i. \tag{38}$$

During our recent investigation, we realize that the LPOE approach, though inadequately explained in [5] and even misunderstood in [18], is essentially equivalent to the ADJ approach (but the error update is different). Since:

$$\begin{aligned}g &= g_{s,1}\cdot e^{\hat{e}_6\theta_1}\cdots g_{5,6}\cdot e^{\hat{e}_6\theta_6}\cdot g_{6,t} \\ &= \prod_{i=1}^{6} e^{\hat{\xi}_i\theta_i}\cdot g_{st},\ \xi_i = Ad_{h_{s,i}}e_6.\end{aligned} \tag{39}$$

we have:

$$\begin{aligned}\xi_i^{k+1} &= Ad_{\prod_{j=1}^{i} e^{\delta\hat{\eta}_j^k}}\xi_i^k \\ &\approx Ad_{e^{\sum_{j=1}^{i}\delta\hat{\eta}_j^k}}\xi_i^k.\end{aligned} \tag{40}$$

The second approximation in (40) holds since $\delta\eta_i^k$'s are considered small. Then (40) is equivalent to (33) via the linear isomorphism:

$$(\delta\eta_1^k,\ldots,\delta\eta_6^k) \mapsto (\delta\eta_1^k, \sum_{j=1}^{2}\delta\eta_j^k,\ldots,\sum_{j=1}^{6}\delta\eta_j^k).$$

IV. A UNIFIED FRAMEWORK FOR COMPARATIVE STUDY OF CALIBRATION ALGORITHMS

A. Parametrization

As shown in Table I, the kinematic parameters K of each model are slightly different.

TABLE I

PARAMETRIZATION OF DIFFERENT MODELING APPROACHES (r: NUMBER OF REVOLUTE JOINTS; t: NUMBER OF PRISMATIC JOINTS).

model	kinematic parameter	total number
D-H	$\{(\theta_i,d_i,\alpha_i,a_i)\}_{i=0}^{7}\in\mathbb{R}^{32}$ $(\alpha_7=a_7=0)$	$4r+2t+6$
POE	$(\xi_1,\ldots,\xi_6;\log(g_{st}))\in se(3)^7$	$6r+3t+6$
LPOE	$(\eta_1,\ldots,\eta_7)\in se(3)^7$	42
ADJ	$(\eta_1,\ldots,\eta_7)\in se(3)^7$	42

According to our explanation in Section II-D, a parametrization $\mathbb{R}^n\to SE(3)/C(\xi)$ of the joint axis configuration space is (locally) complete at a point if and only if the sum of the range $\widehat{W}(\xi)$ of its equivalent Adjoint error and $\mathfrak{c}(\xi)$ equals $se(3)$ (in fact we shall the sum is always direct sum):

$$\widehat{W}(\xi)\oplus\mathfrak{c}(\xi) = se(3). \tag{41}$$

For the D-H parametrization, while $\widehat{W}(\xi)$ is spanned by:

$$\begin{aligned}&\eta_1 = e_6, \eta_2 = e_3, \\ &\eta_3 = Ad_{e^{\hat{e}_6\theta+\hat{e}_3 d}}e_4, \eta_4 = Ad_{e^{\hat{e}_6\theta+\hat{e}_3 d}}e_1.\end{aligned} \tag{42}$$

and $\mathfrak{c}(\xi)$ is spanned by ξ and V_ξ (see (15)), where

$$\xi = Ad_{e^{\hat{e}_6\theta+\hat{e}_3 d}e^{\hat{e}_4\alpha+\hat{e}_1 a}}e_6. \tag{43}$$

In other words, the D-H parametrization is complete at a configuration (θ,d,α,a) if and only if the direct kinematics of a 6-DoF robot manipulator with initial joint axes $(e_6,e_3,e_4,e_1,e_6,e_3)$ is not singular at the configuration $(\theta,d,\alpha,a,\bullet,\bullet)$. It is easy to see that so long as $\alpha\neq 0$ or π, we have:

$$\mathrm{span}\{\eta_1,\ldots,\eta_4;\xi,V_\xi\} = se(3). \tag{44}$$

and the D-H parametrization is complete. When $\alpha=0$ or π, $V_\xi=\pm e_6=\pm\eta_1$ and parametric discontinuity occurs. The application of (44) is geometrically clearer than the approach in [10]. Similarly, for the modified D-H (Hayati) 4-parametrization [20], we shall look at the singularity of the 6-DoF manipulator with initial joint axes $(e_6,e_1,e_4,e_5,e_6,e_3)$ at a configuration $(\theta,a,\alpha,\beta,\bullet,\bullet)$. The singularity set is very complicated and will not be elaborated here.

For the ADJ parametrization, $\widehat{W}(\xi)$ is the image of (34). From standard linear algebra [16]:

$$\text{Im}(I - Ad_{e^{\hat{\xi}\theta}}) = \ker(I - Ad^T_{e^{\hat{\xi}\theta}})^{\perp}. \tag{45}$$

It is trivial to prove that

$$\ker(I - Ad_{e^{\hat{\xi}\theta}}) = \mathfrak{c}(\xi).$$

and that

$$\ker(I - Ad^T_{e^{\hat{\xi}\theta}}) = \text{span}\left\{\begin{bmatrix}\omega\\ v\end{bmatrix}, \begin{bmatrix}0\\ \omega\end{bmatrix}\right\}, \xi = \begin{bmatrix}v\\ \omega\end{bmatrix}.$$

for $\theta \neq N\pi, N \in \mathbb{Z}$. Therefore, $\widehat{W}(\xi) = \mathfrak{c}(\xi)^{\perp}$ and (41) always hold. In other words, the ADJ parametrization is THE most adaptive parametrization against singularity.

For the POE parametrization, we need to find the equivalent Adjoint error $\delta\eta$ such that $Ad_{e^{\delta\hat{\eta}}}\xi = \xi + d\xi$. Since $d\xi$ can be any value and $\xi + d\xi$ can also take any value, the POE parametrization is also complete. However, it is necessarily redundant. To see this, notice that $\delta\eta$ is small, we have:

$$\begin{aligned} Ad_{e^{\delta\hat{\eta}}}\xi &= e^{ad_{\delta\hat{\eta}}}\xi \approx (I + ad_{\delta\hat{\eta}})\xi, \\ &\Rightarrow d\xi = ad_{\delta\eta}\xi = -ad_{\xi}\delta\eta. \end{aligned} \tag{46}$$

In other words, any effective update $d\xi$ will only take the value in the 4-D subspace $\text{Im}(ad_{\xi}) = \mathfrak{c}(\xi)^{\perp}$. The ineffective updates are exactly $\ker ad_{\xi} = \mathfrak{c}(\xi)$. This shows that the claim by [18] that the maximal identifiable parameters for POE parametrization to be $6r + 3r + 6$ is false. Similarly, the LPOE approach is globally complete and its redundancy lies in the non-uniqueness of identified local frames $g_{i-1,i}$'s, as reported in [5].

B. Comparison on Error Jacobian Matrix

Without loss of generality, we shall only consider the case of $6\mathcal{R}$ robot manipulators with end-effector pose measurements. For the D-H approach [19,20], since

$$g = \prod_{i=0}^{7} e^{\hat{e}_6\theta_i + \hat{e}_3 d_i} e^{\hat{e}_4\alpha_i + \hat{e}_1 a_i}. \tag{47}$$

the m-th sample block A^m of the error Jacobian matrix A is given by:

$$\begin{aligned} A^m = [&e_6, e_3, Ad^m_0 e_4, Ad^m_0 e_1; \cdots \\ &; Ad^m_6 e_6, Ad^m_6 e_3] \in \mathbb{R}^{6\times 30}. \end{aligned} \tag{48}$$

Therefore a total of $28M$ exponential maps and $7M$ Adjoint maps are computed to derive the error Jacobian matrix A.

For the POE approach [4,18], since

$$g = e^{\hat{\xi}_1\theta_1} \cdots e^{\hat{\xi}_6\theta_6} e^{\log(g_{st})}. \tag{49}$$

the m-th sample block A^m of the error Jacobian matrix A is given by:

$$A^m = [iAd^m_1; Ad^m_1 iAd^m_2; \ldots; Ad^m_6 iAd^m_{st}] \in \mathbb{R}^{6\times 42}. \tag{50}$$

Therefore a total of $6M$ exponential maps, $6M$ Adjoint maps and integral $7M$ Adjoint maps are computed to derive the error Jacobian matrix A.

For the LPOE approach [5], since

$$g = (\prod_{i=1}^{6} g_{i-1,i} e^{\hat{e}_6\theta_i}) \cdot g_{6,t}. \tag{51}$$

the m-th sample block A^m of the error Jacobian matrix A is given by:

$$A^m = [I; Ad^m_1; \ldots; Ad^m_6] \in \mathbb{R}^{6\times 42}. \tag{52}$$

Therefore a total of $6M$ exponential maps and $6M$ Adjoint maps are computed to derive the error Jacobian matrix A.

Finally, for the ADJ approach, since

$$g = \prod_{i=1}^{6} e^{\hat{\xi}_i\theta_i} \cdot g_{st}. \tag{53}$$

the m-th sample block A^m of the error Jacobian matrix A is given by:

$$\begin{aligned} A^m = [&I - Ad^m_1; Ad^m_1 - Ad^m_2; \ldots \\ &; Ad^m_5 - Ad^m_6; Ad^m_6] \in \mathbb{R}^{6\times 42}. \end{aligned} \tag{54}$$

Therefore a total of $6M$ exponential maps and $6M$ Adjoint maps are computed to derive the error Jacobian matrix A.

1) Redundancy elimination: While POE and LPOE will have ineffective updates due to redundancy, causing the algorithms to stall at improper configurations, the parametrization redundancy in the ADJ approach will cause the error Jacobian matrix to be rank deficient. The kernel is exactly the 12-D subspace $\mathfrak{c}(\xi_1) \oplus \mathfrak{c}(Ad_1\xi_2) \oplus \cdots \oplus \mathfrak{c}(Ad_5\xi_6)$. However, the kernel corresponds to zero singular values and are automatically discarded by the pseudo-inverse subroutine in MATLAB (based on SVD). An alternative solution is to construct a basis $B_i = [\zeta_{i1}, \ldots, \zeta_{i4}] \in \mathbb{R}^{6\times 4}$ for each $\widehat{W}_i(Ad_{i-1}\xi_i) = \mathfrak{c}(Ad_{i-1}\xi_i)^{\perp}$. The differential error kinematics:

$$A^m \cdot \begin{bmatrix}\delta\eta_1\\ \vdots\\ \delta\eta_7\end{bmatrix} = y^m, A^m \in \mathbb{R}^{6\times 42}.$$

becomes:

$$\underbrace{A^m \cdot \begin{bmatrix} B_1 & 0 & \cdots & 0 \\ 0 & \ddots & & \vdots \\ \vdots & 0 & B_6 & 0 \\ 0 & \cdots & 0 & I \end{bmatrix}}_{\tilde{A}^m} \cdot \begin{bmatrix}\delta\tilde{\eta}_1\\ \vdots\\ \delta\tilde{\eta}_7\end{bmatrix} = y^m, \tilde{A}^m \in \mathbb{R}^{6\times 30}.$$

and $\delta\eta_i = B_i\delta\tilde{\eta}_i, i = 1, \ldots, 6$. Although this offers little computational advantage, the imposition of basis matrices B_i's can be used to deal with calibration of constrained manipulators (see the sequel [11]).

2) Computation efficiency: The number of key operations of all four approaches is summarized in Table II. A numerical test on the three classes of operations is conducted in MATLAB, and the time for $1,000$ runs on randomly generated twists is listed in the last row of Table II.

C. Comparison on Error Updates

The error updates of all four approaches are as follows:

D-H $\theta^{k+1}_i = \theta^k_i + \delta\theta^k_i, \ldots$.

POE $\xi^{k+1}_i = \xi^k_i + \delta\xi^k_i, i = 1, \ldots, 7$.

LPOE $g^{k+1}_{i-1,i} = e^{\delta\hat{\eta}^k_i} \cdot g^k_{i-1,i}, i = 1, \ldots, 7$.

ADJ $\xi^{k+1}_i = Ad_{e^{\delta\hat{\eta}^k_i}}\xi^k_i, i = 1, \ldots, 6$;
$g^{k+1}_{st} = e^{\delta\hat{\eta}_{st}} \cdot g_{st}$.

TABLE II
NUMBER/CONFIG. OF KEY OPERATIONS IN COMPUTATION OF ERROR JACOBIAN MATRIX (INCLUDING ERROR UPDATE).

model	$\sharp$ of $e^{(\bullet)}$	$\sharp$ of $Ad_{(\bullet)}$	$\sharp$ of $iAd_{(\bullet)}$
D-H	28	7	-
POE	6	6	7
LPOE	6 (13)	6	-
ADJ	6 (13)	6 (12)	-
$sec/1K$	0.047	0.021	0.19

The overall number of operations is listed in the bracket in Table II. Since roughly, $e^{(\bullet)}$ and $iAd_{(\bullet)}$ take twice and ten times as long as $Ad_{(\bullet)}$, the efficiency of LPOE and ADJ approach should be comparable and much higher than the D-H and POE approach, where the POE is the worst.

D. Structure clarity of error Jacobian matrix

Other than the aspects mentioned in the above, we would also like to point out the following fact. Although the ADJ approach takes slightly more $Ad_{(\bullet)}$ operations than the LPOE approach, the structure clarity of the corresponding error Jacobian matrix throws much light on the problem of elimination of parametrization redundancy, identifiability, dealing with constraints, optimal selection of measurement configurations with different calibration indices. We have achieved some exciting results and conclusions in some of these issues, and will report them in the sequel to this paper.

V. CONCLUSIONS

In this paper, we have proposed a unified geometric framework for the comparison of robustness and efficiency of robot kinematic calibration algorithms. We proposed a novel error modeling method along with a novel calibration algorithm known as the *Adjoint error* algorithm. This algorithm is essentially equivalent to Chen's local POE approach. However, our new formulation offers a much deeper geometric understanding, and also a natural way to eliminate the redundancy in the parametrization. The elimination of redundancy is essentially choosing an adaptive basis for the complement of the joint subalgebra (Adjoint errors that fix the joint axis), where there is no strings attached to the choice of the basis. In fact, any minimal parametrization boil down to choosing a basis. In comparison, all other conventions require at least two sets of 4 (2 for prismatic joint) minimal parameters to achieve completeness. Those with more than 4 (2) parameters achieves parametric continuity when the error basis together with the joint subalgebra generates the whole twist space $se(3)$. The issue of redundancy elimination becomes much clearer under our framework. Therefore, our framework offers not only a reconciliation between major kinematic modeling conventions, but also a unified method to consider the three rules of thumb for choosing appropriate parametrization.

An efficient estimation of the algorithm efficiency can be achieved by counting the number of Adjoint maps and integral Adjoint maps needed for the derivation of the error Jacobian matrix, the latter being roughly twice slower than the former. The POE and D-H algorithms should be much slower than the local POE and Adjoint error algorithms.

To summarize the essential contribution of our work: our geometric framework offers a test bed for any parametric kinematic calibration algorithms, which can not only quickly determine their efficiency (by simply counting the number of Adjoint maps and/or integral Adjoint maps), but also analyze their pros and cons over parametrization issues, identifiability, imposing constraints, etc. The latter two, along with other equally important issues are treated in the sequel to this paper.

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